Determinants

Last Time: Computational Introduction to Determinants.

↳ Cofactor Expansion Formula
  (AKA Laplace Expansion Formula)

↳ Many Examples...

↳ Determinants of Elementary Matrices. ✶

Recall: $\det(P_{i,j}) = -1 \qquad (i \neq j)$

✶ $\det(M_i(k)) = k$ ✶

$\det(A_{i,j}(k)) = 1$

Def^n: The $n \times n$ determinant function is the function $\det : M_{n\times n} \to \mathbb{R}$ satisfying these conditions:

✶ ① $\det(\ell_1, \ell_2, \cdots, k\ell_i + \ell_j, \cdots, \ell_n) = \det(\ell_1, \ell_2, \cdots, \ell_n)$.

② $\det(\ell_1, \ell_2, \cdots, \ell_{i-1}, \ell_j, \ell_{i+1}, \cdots, \ell_{j-1}, \ell_i, \ell_{j+1}, \cdots, \ell_n)$
$= -\det(\ell_1, \ell_2, \cdots, \ell_n)$.

③ $\det(\ell_1, \ell_2, \cdots, k\ell_i, \cdots, \ell_n) = k\det(\ell_1, \cdots, \ell_n)$

④ $\det(I_n) = 1$.

__NB:__ The above properties are indeed satisfied by the cofactor expansion formula... (they're a lot nasty to prove...)

__Point:__ determinants are computable using row operations :)

__Ex:__ Compute $\det\begin{bmatrix} 1 & 0 & -1 & 3 \\ 3 & 0 & 1 & -5 \\ 1 & 2 & 3 & 5 \\ 5 & 10 & 15 & 20 \end{bmatrix}$.

__Sol:__

$$\det\begin{bmatrix} 1 & 0 & -1 & 3 \\ 3 & 0 & 1 & -5 \\ 1 & 2 & 3 & 5 \\ 5 & 10 & 15 & 20 \end{bmatrix} \quad \star$$

$$= \underline{\underline{5}} \det\begin{bmatrix} 1 & 0 & -1 & 3 \\ 3 & 0 & 1 & -5 \\ 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

subtracting multiples of r_1 from r_2, r_3, r_4

$$= 5 \det\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 0 & 4 & -14 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 4 & 1 \end{bmatrix}$$

swap

$$= 5\cdot(-1) \det\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 4 & -14 \\ 0 & 2 & 4 & 1 \end{bmatrix}$$

$$= -5 \det\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 4 & -14 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

row-echelon form, "eliminate upwards"

$$= -5 \det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

apply scaling rule multiple times...

$$= -5\,(2)(4)(-1)\det(I_4) = -5\cdot 2\cdot 4\cdot -1\cdot 1 = \underline{\underline{40}} \quad \boxtimes$$

__Exercise:__ Compute $\det(M)$ above via cofactor expansion...

Ex: Compute $\det\begin{bmatrix}-1 & 1 & 5\\ 5 & 8 & 6\\ 3 & 9 & -7\end{bmatrix}$.

Sol: $\det\begin{bmatrix}-1 & 1 & 5\\ 5 & 8 & 6\\ 3 & 9 & -7\end{bmatrix} = \det\begin{bmatrix}-1 & 1 & 5\\ 0 & 13 & 31\\ 0 & 12 & 8\end{bmatrix} \leftarrow$

$= 4\det\begin{bmatrix}-1 & 1 & 5\\ 0 & 13 & 31\\ 0 & 3 & 2\end{bmatrix}$

$= 4\cdot\frac{1}{3}\det\begin{bmatrix}-1 & 1 & 5\\ 0 & \boxed{39} & 93\\ 0 & 3 & 2\end{bmatrix}\begin{matrix}\\ \leftarrow\\ \leftarrow\end{matrix}$

Echelon form.

$= \frac{4}{3}\det\begin{bmatrix}-1 & 1 & 5\\ 0 & \boxed{0} & 67\\ 0 & 3 & 2\end{bmatrix} = -\frac{4}{3}\det\begin{bmatrix}-1 & 1 & 5\\ 0 & 3 & 2\\ 0 & 0 & 67\end{bmatrix}$

$= -\frac{4}{3}\begin{bmatrix}-1 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 67\end{bmatrix} = -\frac{4}{3}(-1)(3)(67)\det\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

$= -\frac{4}{3}(-1)(3)(67)\cdot 1 = 268$ ✓ $\boxtimes$

Sol 2 (Via Cofactor Expansion):

$\det\begin{bmatrix}\overset{+}{-1} & \overset{-}{1} & \overset{+}{5}\\ 5 & 8 & 6\\ 3 & 9 & -7\end{bmatrix} = -\det\begin{bmatrix}8 & 6\\ 9 & -7\end{bmatrix} \overset{\downarrow}{-} \det\begin{bmatrix}5 & 6\\ 3 & -7\end{bmatrix} + 5\det\begin{bmatrix}5 & 8\\ 3 & 9\end{bmatrix}$

$= -(8\cdot -7 - 6\cdot 9) - (5\cdot -7 - 6\cdot 3) + 5(5\cdot 9 - 3\cdot 8)$

$= -(-56 - 54) - (-35 - 18) + 5(45 - 24)$

$= 110 + 53 + 5(21) = 163 + 105 = 268$ $\boxtimes$

Prop: The cofactor Expansion Formula and the properties of det given at the beginning of the lecture determine the same quantity for every $n \times n$ matrix. In particular, the determinant function is given by either.

Ex: Compute $\det \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 3 \\ 0 & 3 & 4 \end{bmatrix}$

Sol:

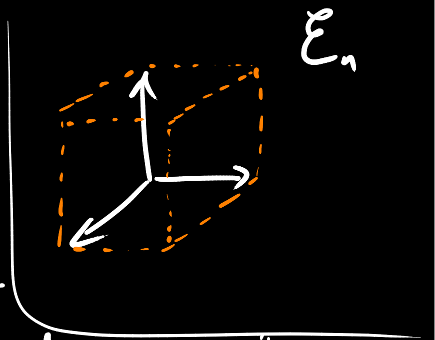
$$\det \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 3 \\ 0 & 3 & 4 \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 4 \\ 0 & 3 & 4 \end{bmatrix}$$

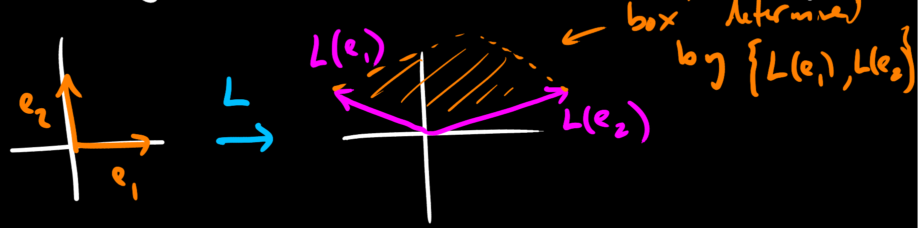
$$= \det \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\boxed{/\!/}$

Prop: Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Let $[L]$ be the matrix of L with respect to the standard basis on $\mathbb{R}^n$ (i.e. $[L] = [L(e_1) \mid L(e_2) \mid \cdots \mid L(e_n)]$). The determinant $\det[L]$ is the "signed volume" of the "box" determined by $\{L(e_1), L(e_2), \ldots, L(e_n)\}$.



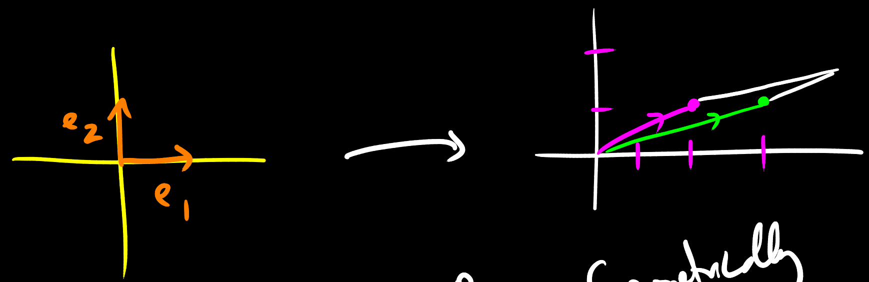
Picture in $\mathbb{R}^2$:

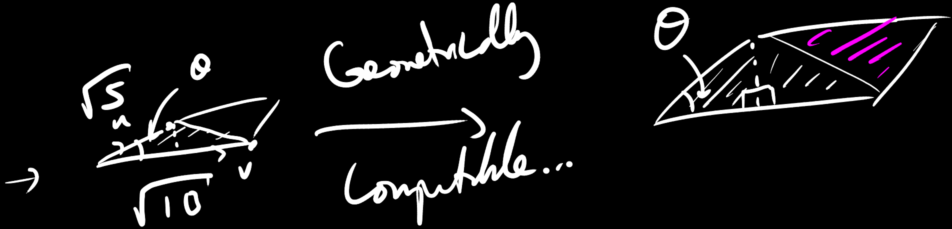


NB: Proof omitted for time, see Hefferon... $\boxed{\nabla/\Delta}$

Ex: Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ have matrix $\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$.

"Box" = "parallelopiped".





$$\det \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = 2-3 = -1 \leftarrow \therefore \text{Area} = |-1| = 1. \ \boxed{\nabla/\Delta}$$

Cor: The determinant is multiplicative. I.E. For $A, B \in M_{n\times n}$ we have $\det(AB) = \det(A)\det(B)$.

Pf: A and B determine two linear transformations $\mathbb{R}^n \to \mathbb{R}^n$. The product is the matrix of their composition. Then $\det(AB) \overset{*}{=}$ volume of the parallelopiped determined by $AB(\mathcal{E}_n) = A(B\mathcal{E}_n)$. So we see

$$\det(AB) \overset{*}{=} \det(A) \cdot \text{volume}(\text{parallelopiped given by } B\mathcal{E}_n)$$
$$\overset{*}{=} \det(A)\det(B). \quad \smile \quad \boxed{\nabla/\Delta}$$

proposition $\smile$

NB: This isn't particularly surprising... The definition of the determinant given today encodes the conditions

$$\det(\text{Product of elem. mats}) = \text{prod}(\text{dets of the elem mats}) \smile.$$

Cor: Suppose A is invertible. Then $\det(A^{-1}) = \frac{1}{\det(A)}$.

pf: If A is invertible, then $I_n = A^{-1}A$,

so $1 = \det(I_n) = \det(A^{-1}A) = \underline{\det(A^{-1}) \cdot \det(A)}$.

Hence dividing both sides by $\det(A)$ yields result. $\boxtimes$

Exercise; Check for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$ directly ...

Cor: Let A be an $n \times n$ matrix. Then $\det(A) \neq 0$ if and only if A is invertible.

pf: If A is invertible, $\det(A^{-1}) \cdot \det(A) \neq 0$, so $\det(A) \neq 0$.

If $\det(A) \neq 0$, then $L_A : \mathbb{R}^n \to \mathbb{R}^n$ determined by A takes the parallelopiped of $\mathcal{E}_n$ to a parallelopiped of nonzero volume. Moreover, if $L_A(x) = 0$ for $x \neq 0$, then extending $\{x\}$ to a basis of $\mathbb{R}^n$ would yield a parallelopiped which maps under L_A to a zero-volume parallelopiped, hence contradicting the theorem. $\boxtimes$